Signal and Systems

By

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كلية الهندســـة بشبرا

FACULTY OF ENGINEERING- SHOUBRA

Lecture (6)

Course Title: **Signal and Systems**

Course Code: **ELE 115**

Contact Hours: **5.**

= [2 Lect. + 2 Tut + 1 Lab]

Assessment:

Final Exam: 75.

Midterm: ??.

Year Work & Quizzes: 50.

Experimental/Oral: 25.

Textbook:

1- E. W. Kamen and B. S. Heck, Fundamentals of Signals and Systems Using the Web and MATLAB, 3rd ed., Pearson Hihgher Education, 2006.
2- Benjamin C. Kuo " Automatic control systems" 9th ed., John Wiley & Sons, Inc. 2010.
3- Katsuhiko Ogata, "Modern Control Engineering", 4th Edition, 2001.

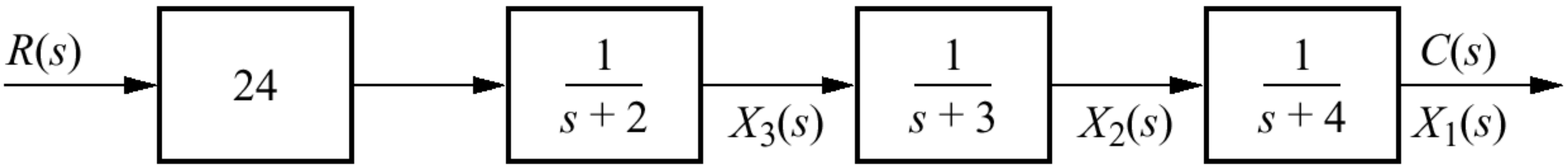
Course Description

- Introduction, fundamentals and basic properties of signals and systems, definition of open loop and closed loop systems, mathematical models of physical systems (mechanical, electrical, electromechanical systems …), control system components, block diagram simplification, signal flow graph, state variable models, Z-Transform and its properties, solving difference equations, pulse transfer function of discrete system, Fourier transforms, continuous and discrete signal analysis, transient response of first and second order control systems, real life applications such as analog and digital filters, introduction to basics of digital signal processor (DSP) and its features and capabilities of commercial applications.

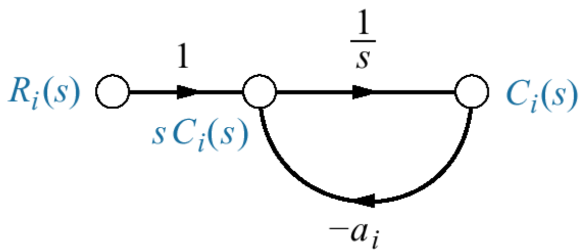
Mathematical modeling of linear dynamic systems & transfer function
Block Diagram Fundamentals
&
Reduction Techniques

Decomposition

Alternate Representation: Cascade Form



$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$



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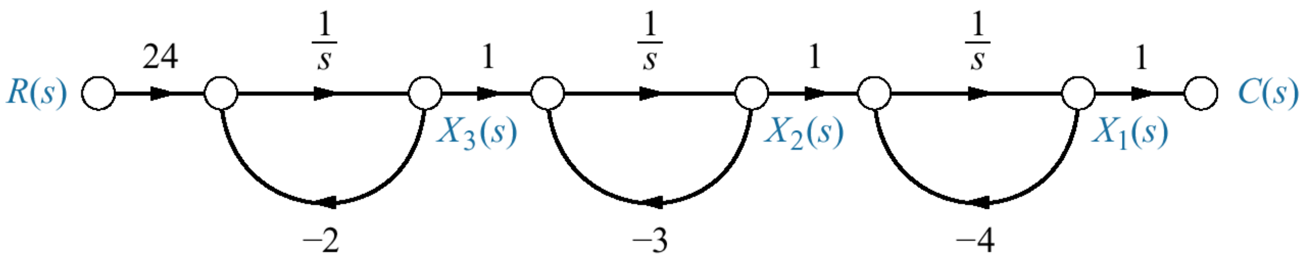

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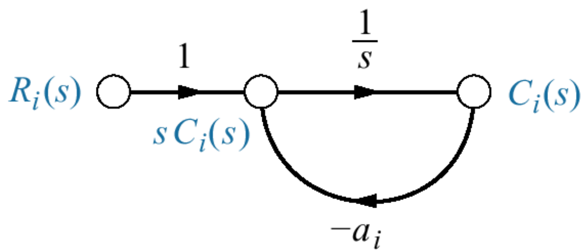
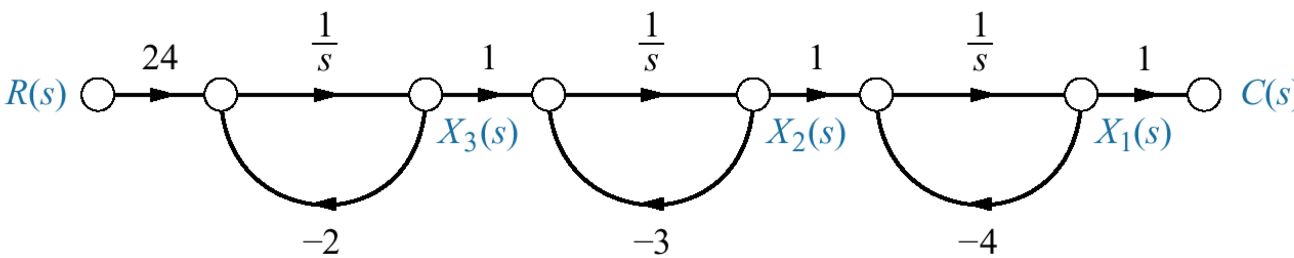
Alternate Representation: Cascade Form



(*a*)



(*b*)

$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = -3x_2 + x_3$$
$$\dot{x}_3 = -2x_3 + 24r$$
$$y = c(t) = x_1$$

$$\dot{X} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

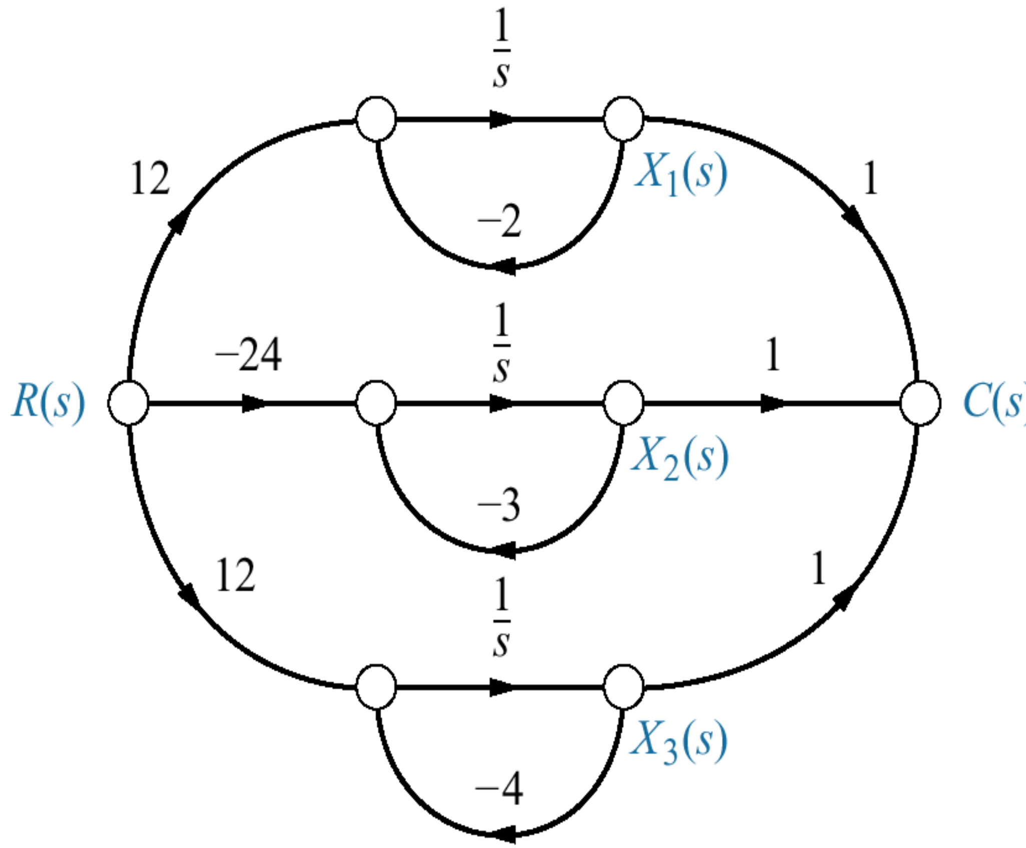
Alternate Representation: Parallel Form

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$

$$\dot{x}_1 = -2x_1 \qquad\qquad\qquad\qquad +12r$$
$$\dot{x}_2 = \qquad -3x_2 \qquad\qquad -24r$$
$$\dot{x}_3 = \qquad\qquad\qquad -4x_3 +12r$$
$$y = c(t) = x_1 + x_2 + x_3$$

$$\dot{X} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} X + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} X$$



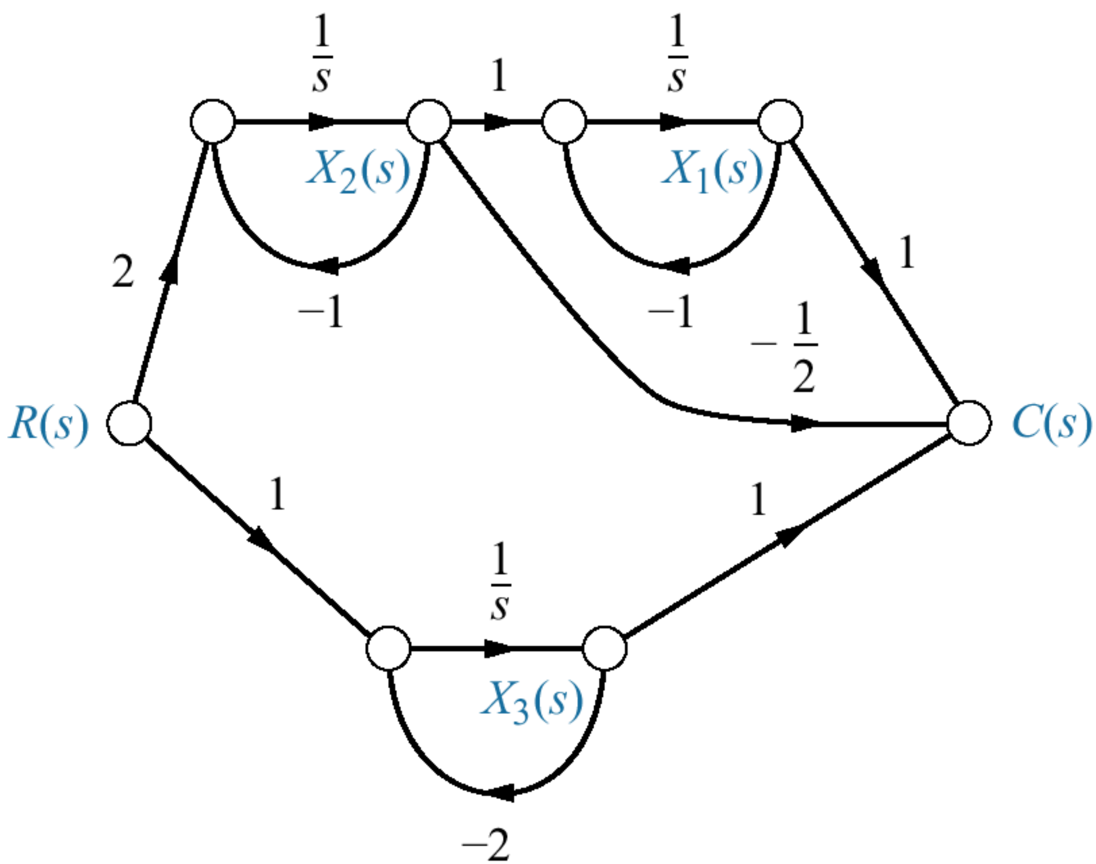
Alternate Representation: Parallel Form Repeated roots

$$\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)} = \frac{2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}$$

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = x_2 + 2r$$
$$\dot{x}_3 = -2x_3 + r$$
$$y = c(t) = x_1 - 1/2x_2 + x_3$$

$$\dot{X} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$$

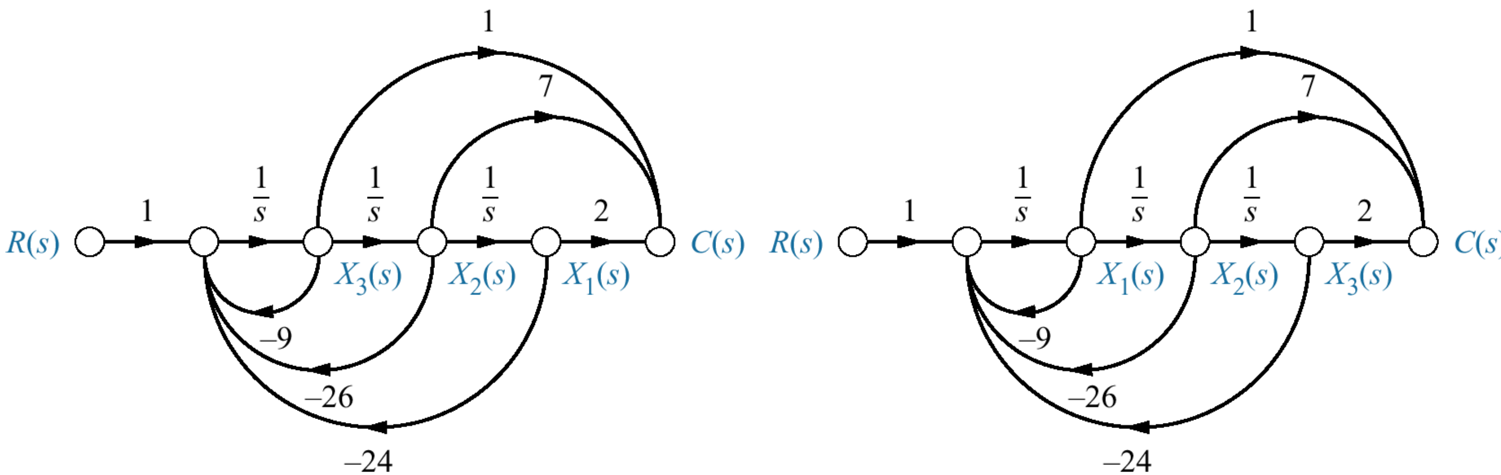
$$y = \begin{bmatrix} 1 & -1/2 & 1 \end{bmatrix} X$$



Alternate Representation: controller canonical form

$G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

This form is obtained from the phase-variable form simply by ordering the phase variable in reverse order



(a) (b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Alternate Representation: controller canonical form

System matrices that contain the coefficients of the characteristic polynomial are called *companion matrices* to the characteristic polynomial.

Phase-variable form result in lower companion matrix

Controller canonical form results in upper companion matrix

Alternate Representation: observer canonical form

Observer canonical form so named for its use in the design of observers

$G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

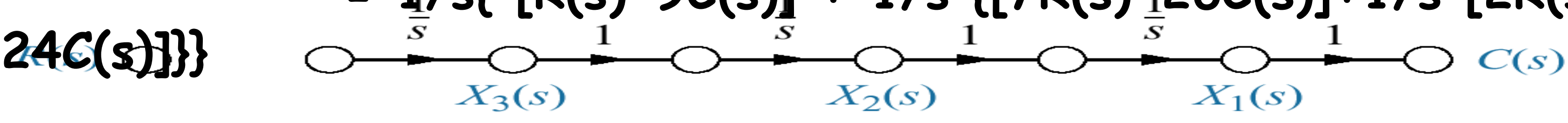
$= (1/s+7/s^2 +2/s^3)/(1+9/s+26/s^2 +24/s^3)$

Cross multiplying

$(1/s+7/s^2 +2/s^3)R(s) = (1+9/s+26/s^2 +24/s^3) C(s)$

And $C(s) = 1/s[R(s)-9C(s)] +1/s^2[7R(s)-26C(s)]+1/s^3[2R(s)-24C(s)]$

$= 1/s\{ [R(s)-9C(s)] + 1/s \{[7R(s)-26C(s)]+1/s [2R(s)-24C(s)]\}\}$



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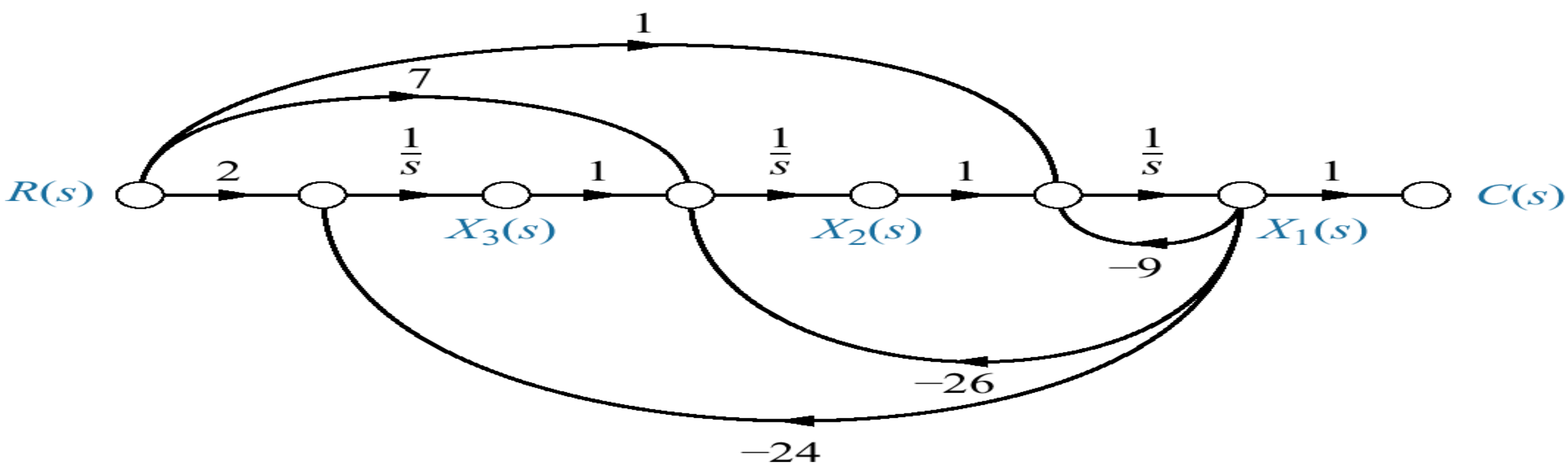


(b)

Alternate Representation: observer canonical form

$$\dot{x}_1 = -9x_1 + x_2 + r$$

$$\dot{x}_2 = -26x_1 + x_3 + 7r$$

$$\dot{x}_3 = -24x_1 + 2r$$

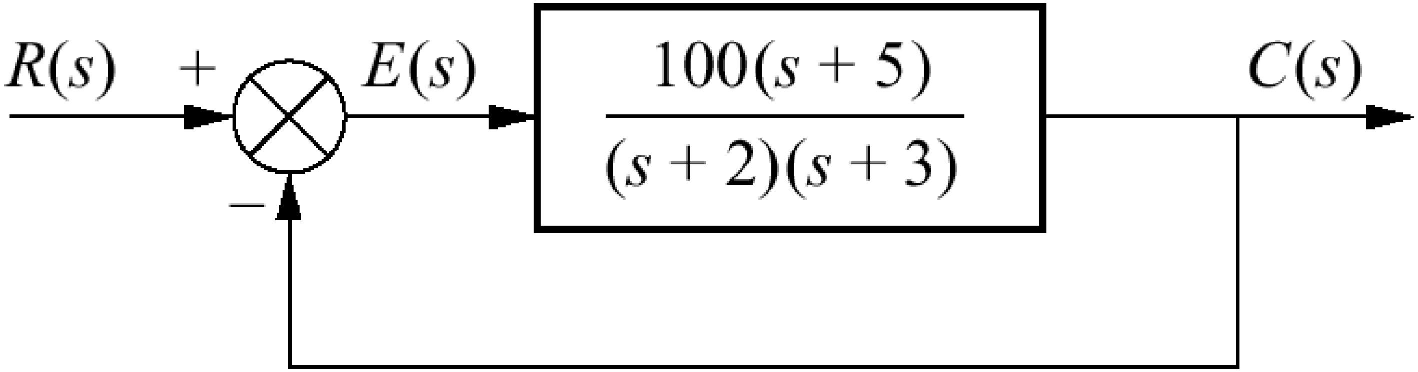
$$y = c(t) = x_1$$

$$\dot{X} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

Note that the observer form has A matrix that is transpose of the controller canonical form, B vector is the transpose of the controller C vector, and C vector is the transpose of the controller B vector. The 2 forms are called duals.

Feedback Control System Example



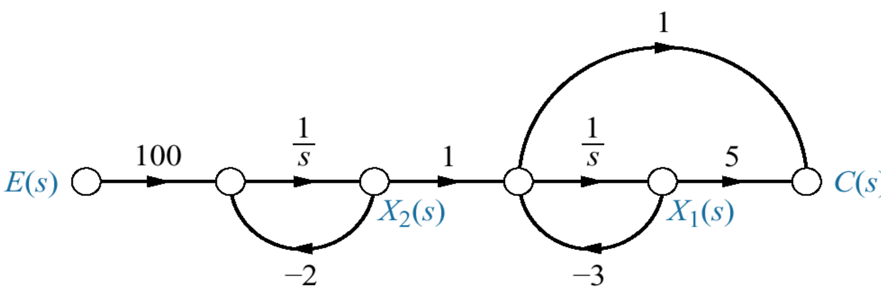
Problem Represent the feedback control system shown in state space. Model the forward transfer function in cascade form.

Solution first we model the forward transfer function as in (a), Second we add the feedback and input paths as shown in (b) complete system. Write state equations

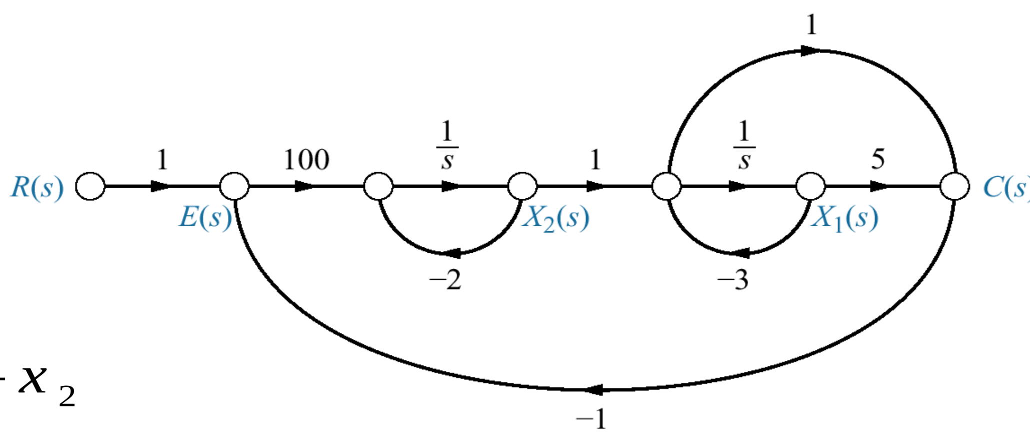
$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = -2x_2 + 100(r - c)$$

$$\text{but } c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2$$



(a)



(b)

Feedback Control System Example

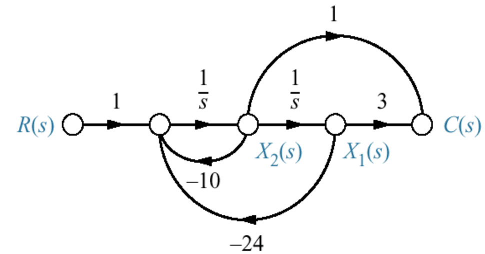
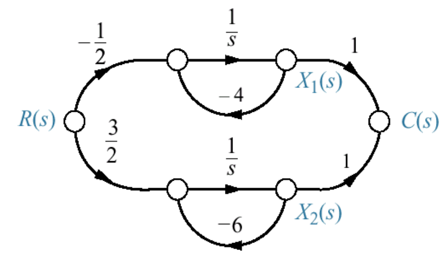
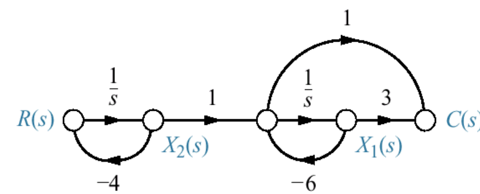
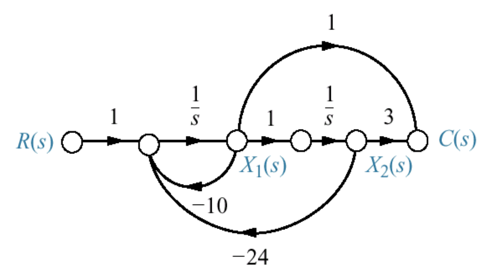
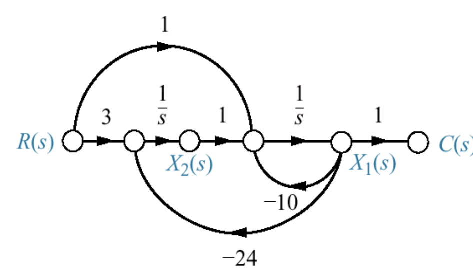
$$\dot{x}_1 = -3x_1 + x_2$$
$$\dot{x}_2 = -200x_1 - 102x_2 + 100r$$

$$y = c(t) = 2x_1 + x_2$$

$$\dot{X} = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix} X + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} X$$

State-space forms for

$C(s)/R(s) = (s+3)/[(s+4)(s+6)]$.
Note: $y = c(t)$

Form	Transfer Function	Signal-Flow Diagram	State Equations
Phase variable	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [3 \quad 1]\mathbf{x}$
Parallel	$\frac{-1/2}{(s+4)} + \frac{3/2}{s+6}$		$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$ $y = [1 \quad 1]\mathbf{x}$
Cascade	$\frac{1}{(s+4)} * \frac{(s+3)}{(s+6)}$		$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [-3 \quad 1]\mathbf{x}$
Controller canonical	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$ $y = [1 \quad 3]\mathbf{x}$
Observer canonical	$\frac{\frac{1}{s} + \frac{3}{s^2}}{1 + \frac{10}{s} + \frac{24}{s^2}}$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$ $y = [1 \quad 0]\mathbf{x}$

Time Domain Analysis

Introduction

In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.

It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.

Usually, the input signals to control systems are not known fully ahead of time.

It is therefore difficult to express the actual input signals mathematically by simple equations.

Standard Test Signals

The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.

The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.

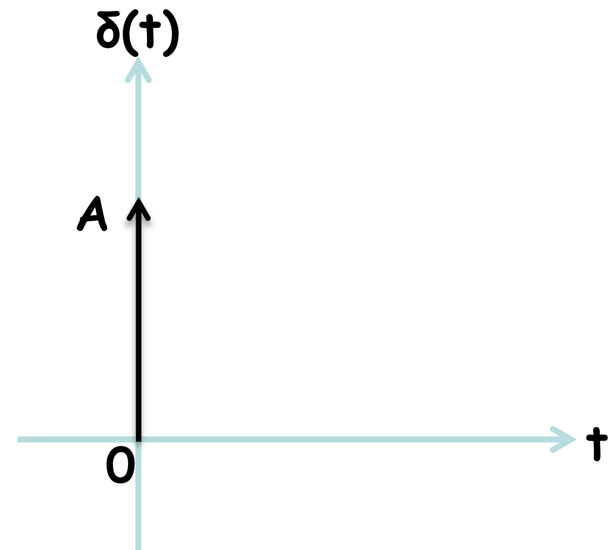
The other standard signal of great importance is a sinusoidal signal.

Standard Test Signals

Impulse signal

The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

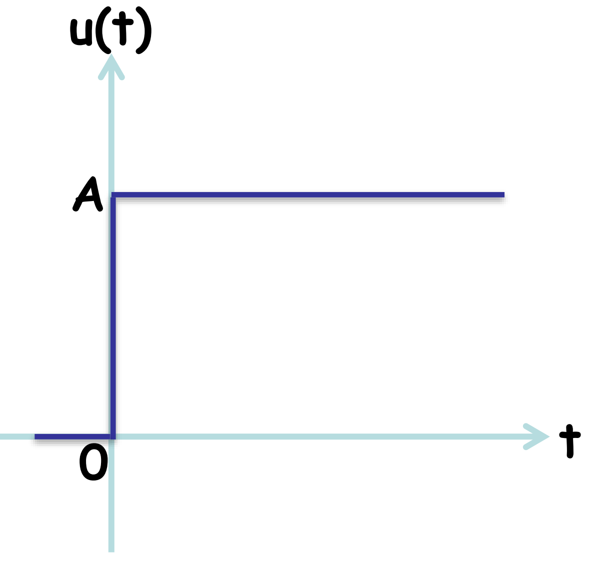


If A=1, the impulse signal is called unit impulse signal.

Standard Test Signals

Step signal

The step signal imitate the sudden change characteristic of actual input signal.



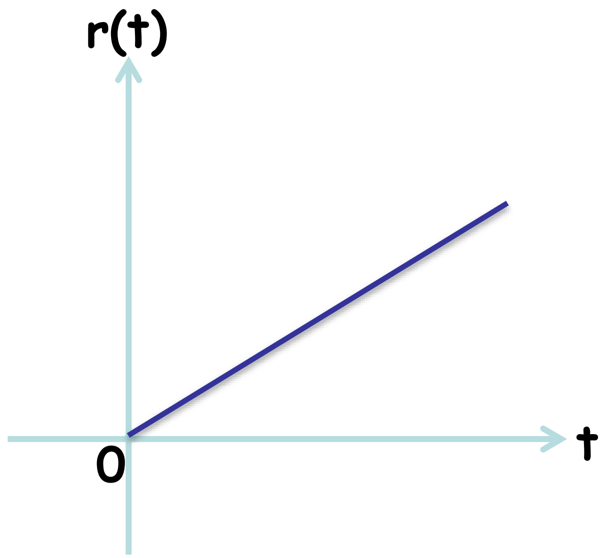
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

If A=1, the step signal is called unit step signal

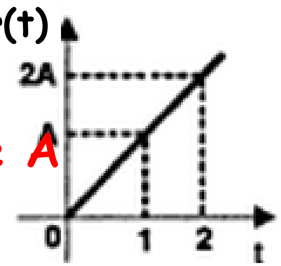
Standard Test Signals

Ramp signal

The ramp signal imitate the constant velocity characteristic of actual input signal.

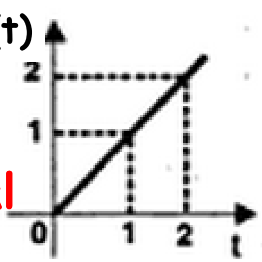


$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$



ramp signal with slope *A*

If *A=1*, the ramp signal is called unit ramp signal

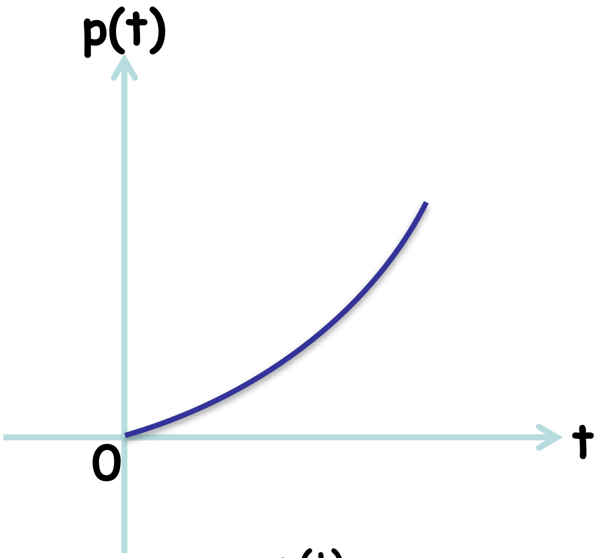


unit ramp signal

Standard Test Signals

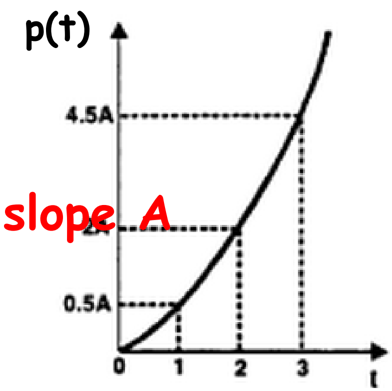
Parabolic signal

The parabolic signal imitate the constant acceleration characteristic of actual input signal.

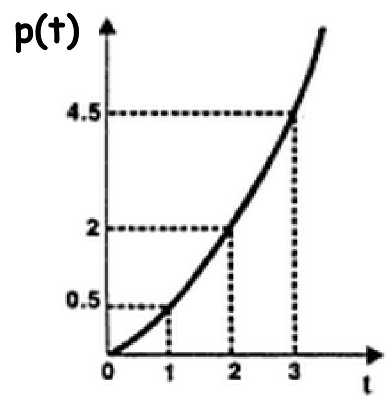


$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

parabolic signal with slope A



If *A=1*, the parabolic signal is called unit parabolic signal.



Unit parabolic signa

Relation between standard Test Signals

Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Parabolic

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Each signal is the integral $\int$ of the one above it, and each signal is the derivative $\dfrac{d}{dt}$ of the one below it.

Laplace Transform of Test Signals

Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

Laplace Transform of Test Signals

Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

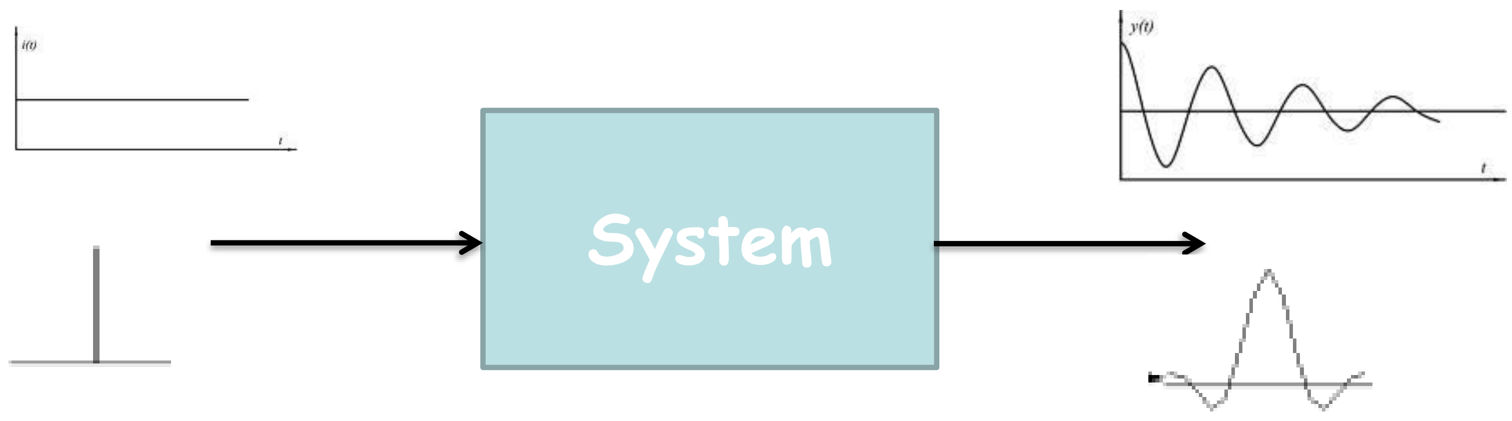
Parabolic

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

Time Response of Control Systems

- Time response of a dynamic system response to an input expressed as a function of time.



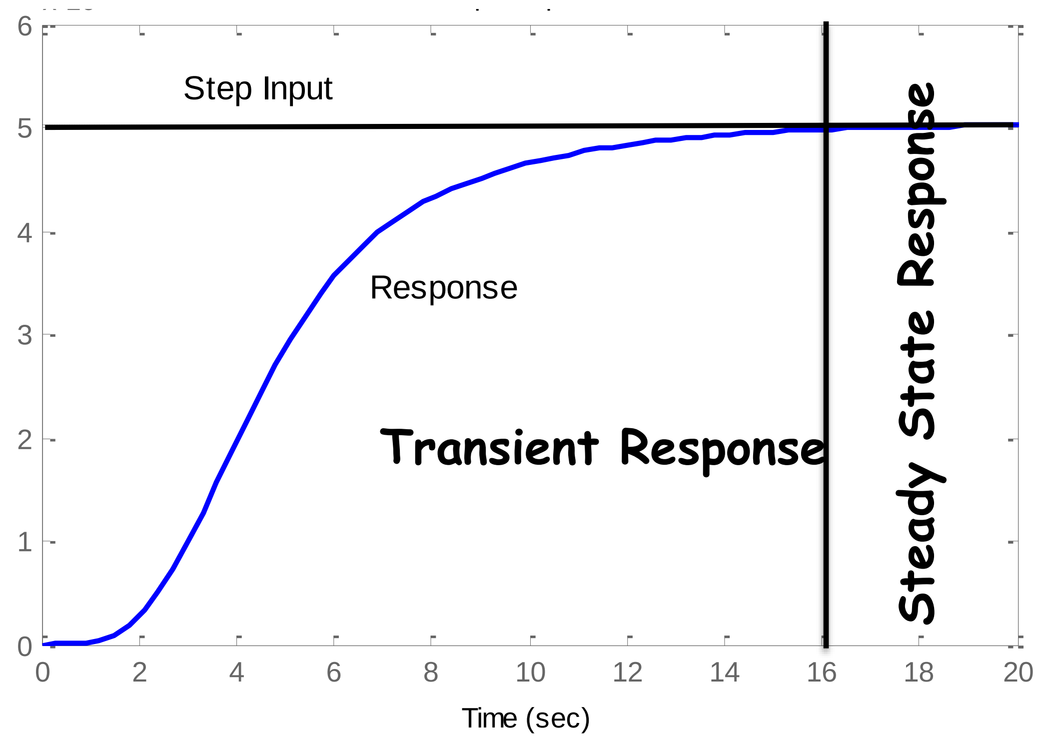
- The time response of any system has two components
  - Transient response
  - Steady-state response.

Time Response of Control Systems

- **When the response of the system is changed from equilibrium it takes some time to settle down.**

- **This is called transient response.**

- **The response of the system after the transient response is called steady state response.**



Time Response of Control Systems

- **Transient response depend upon the system poles only and not on the type of input.**
- **It is therefore sufficient to analyze the transient response using a step input.**
- **The steady-state response depends on system dynamics and the input quantity.**
- **It is then examined using different test signals by final value theorem.**

Introduction

The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

Where **K** is the D.C gain and **T** is the time constant of the system.

Time constant is a measure of how quickly a 1st order system responds to a unit step input.

D.C Gain of the system is ratio between the input signal and the steady state value of output.

Introduction

The first order system given below.

$$G(s) = \frac{10}{5s+1}$$

- D.C gain is 10 and time constant is 5 seconds.

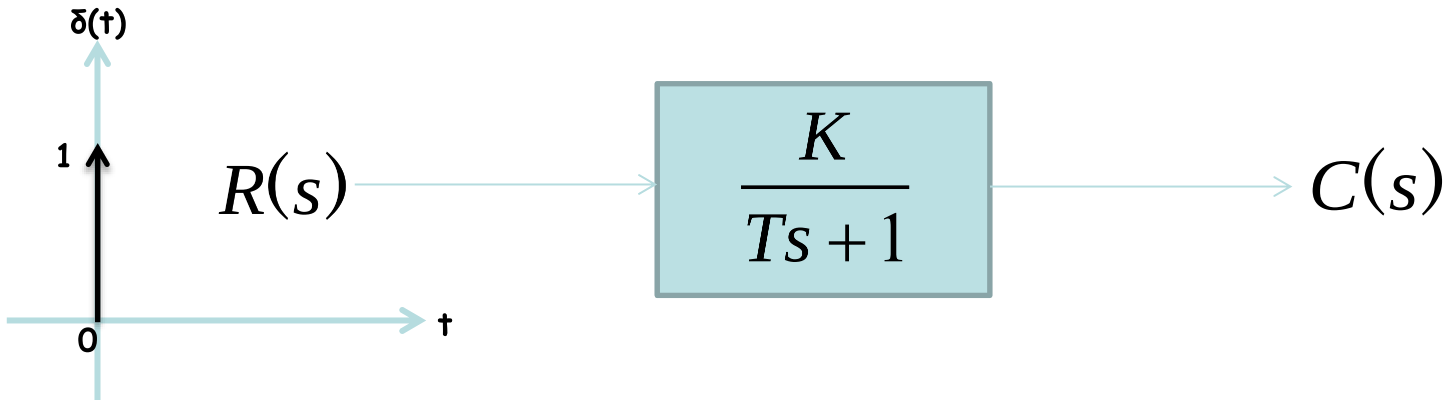
- For the following system

$$G(s) = \frac{6}{s+2} \qquad = \frac{6/2}{1/2s+1}$$

- **D.C Gain of the system is 6/2 and time constant is 1/2 seconds.**

Impulse Response of 1st Order System

Consider the following 1st order system



$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts+1}$$

Re-arrange following equation as

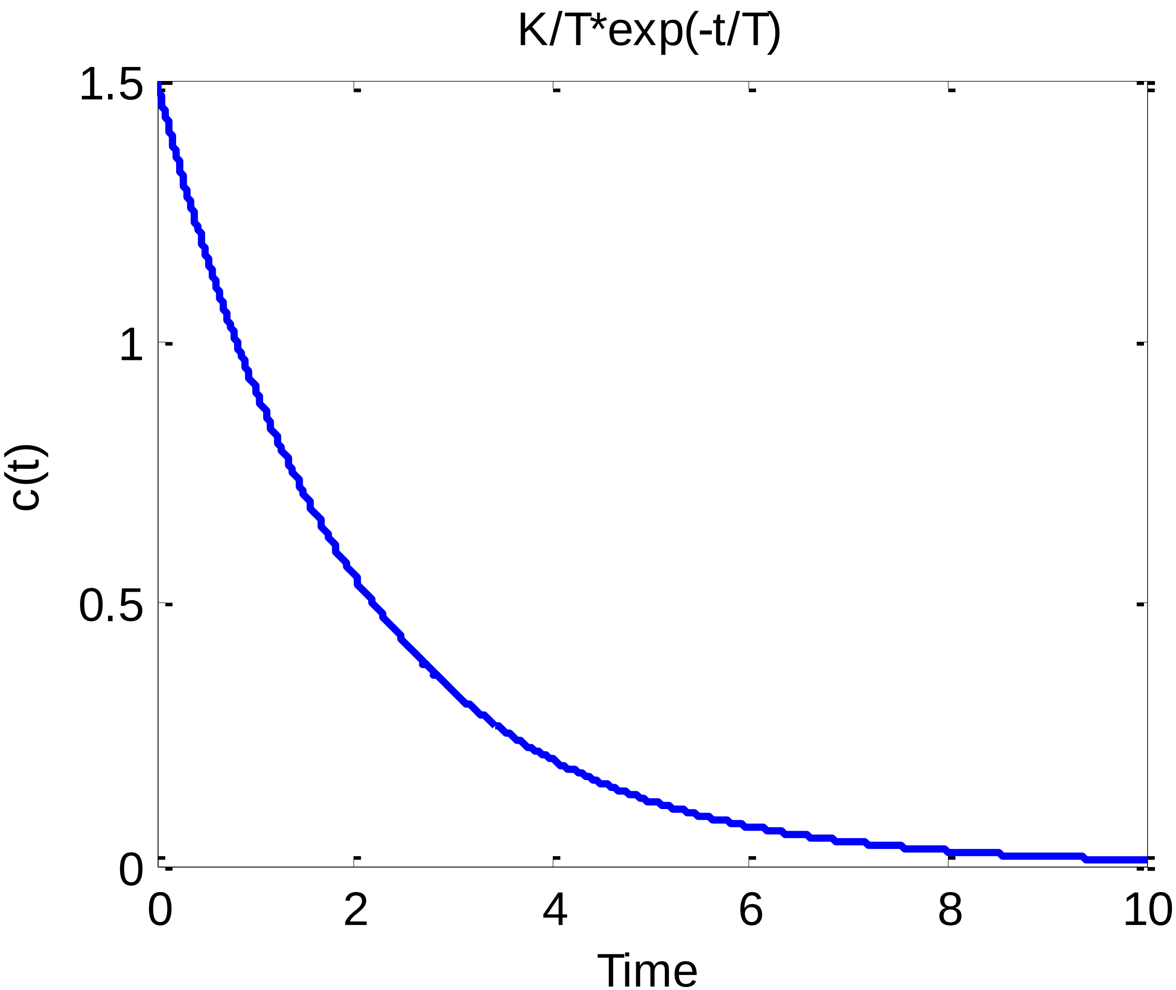
$$C(s) = \frac{K/T}{s+1/T}$$

- **In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.**

$$L^{-1}\left(\frac{C}{s+a}\right) = Ce^{-at} \qquad c(t) = \frac{K}{T}e^{-t/T}$$

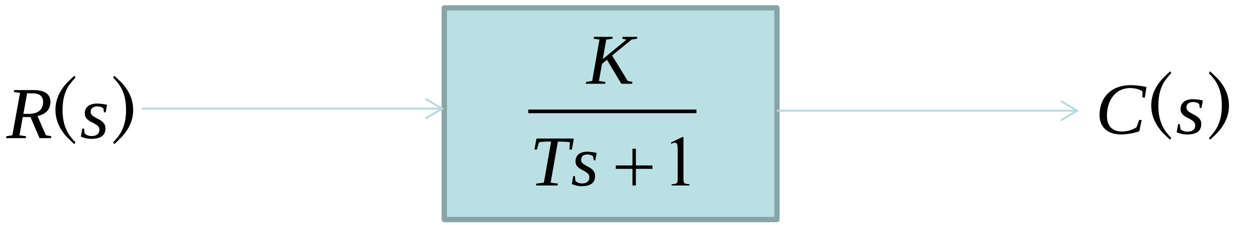
Impulse Response of 1st Order System

- **If K=3 and T=2s then** $c(t) = \frac{K}{T} e^{-t/T}$



Step Response of 1st Order System

Consider the following 1st order system



$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- **In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion.**

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

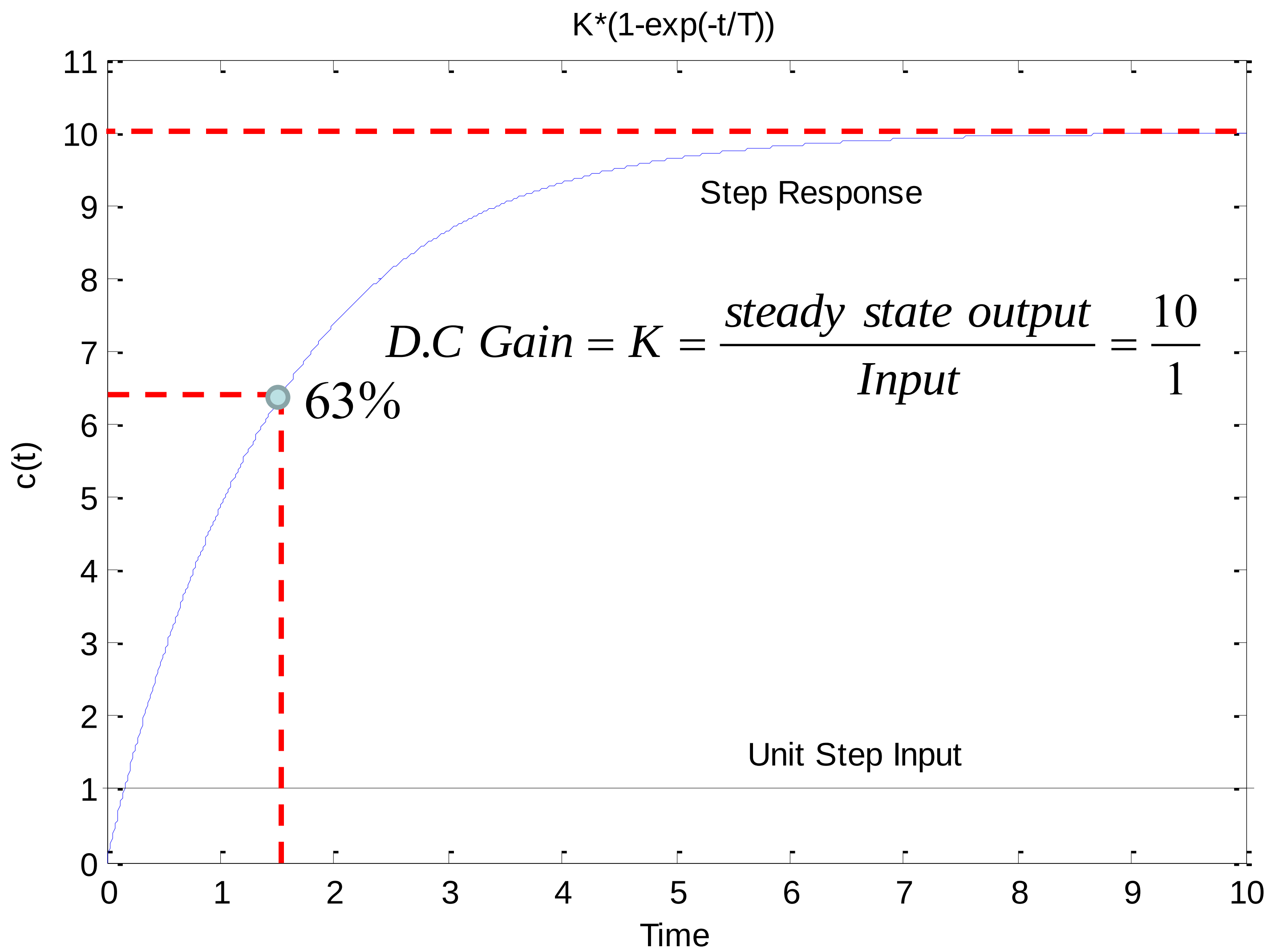
$$c(t) = K\left(1 - e^{-t/T}\right)$$

- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

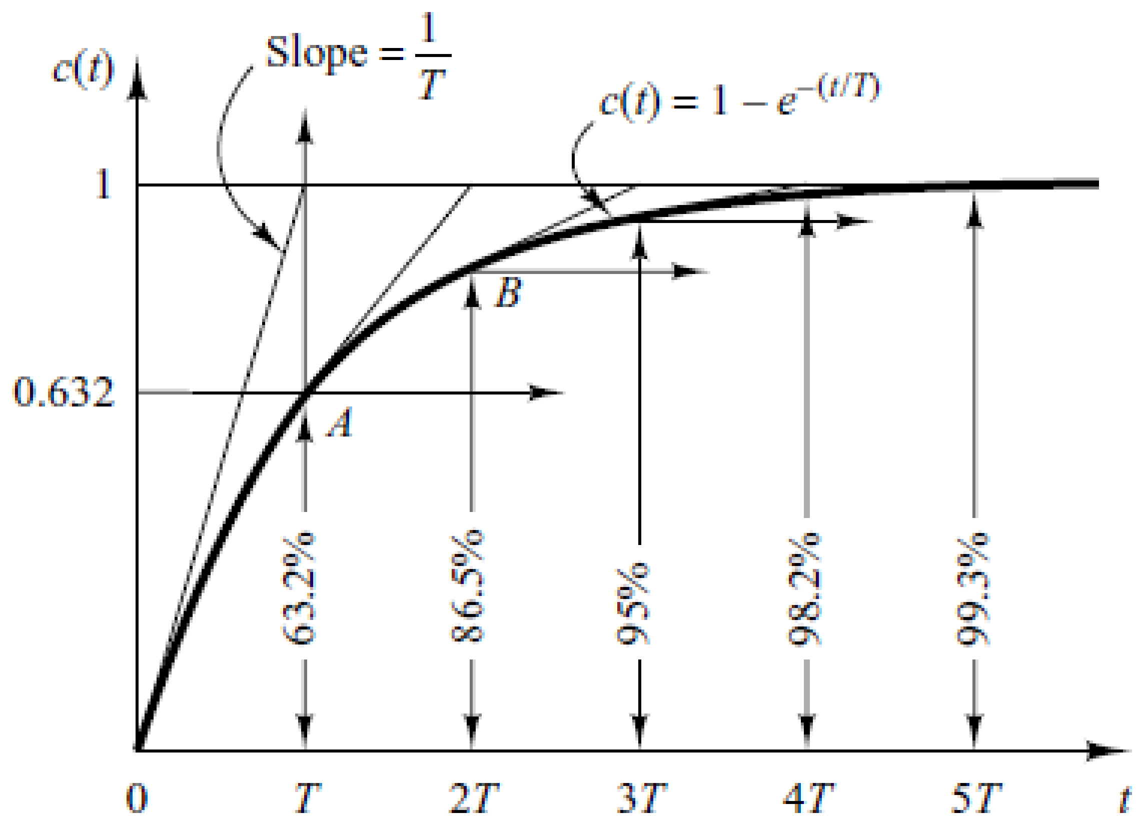
Step Response of 1st Order System

- If K=10 and T=1.5s then $c(t) = K\left(1 - e^{-t/T}\right)$



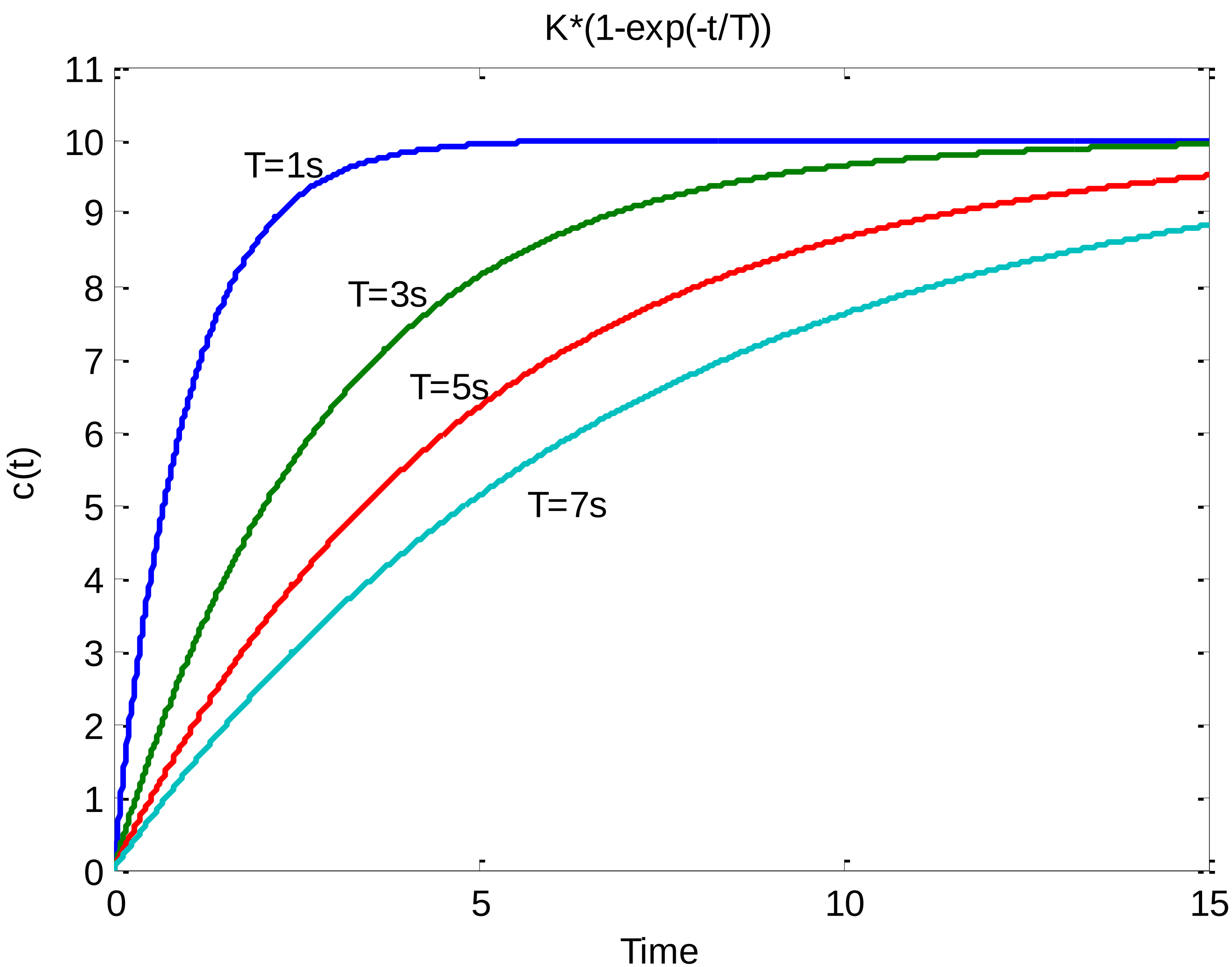
Step Response of 1st order System

System takes five time constants to reach its final value.



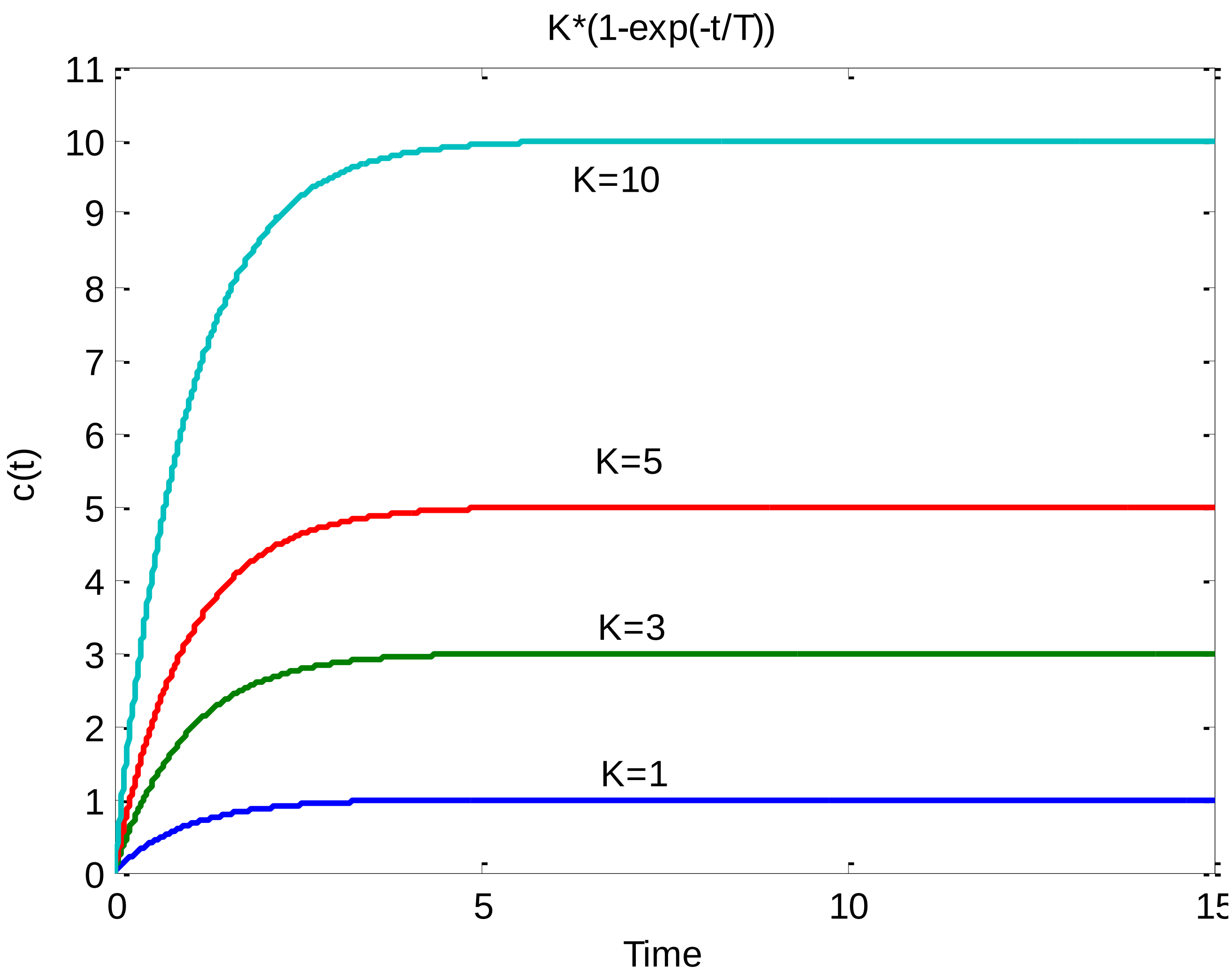
Step Response of 1st Order System

- If K=10 and T=1, 3, 5, 7 $c(t) = K\left(1 - e^{-t/T}\right)$



Step Response of 1st Order System

- If K=1, 3, 5, 10 and T=1 $c(t) = K\left(1 - e^{-t/T}\right)$



Relation Between Step and impulse response

The step response of the first order system is

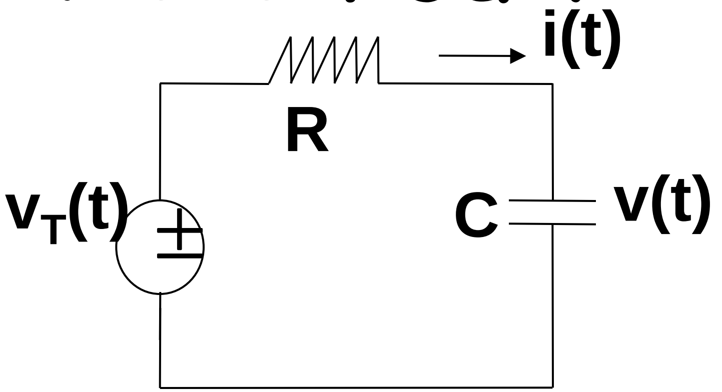
$$c(t) = K\left(1 - e^{-t/T}\right) = K - Ke^{-t/T}$$

Differentiating c(t) with respect to t yields

$$\frac{dc(t)}{dt} = \frac{d}{dt}\left(K - Ke^{-t/T}\right)$$

$$\frac{dc(t)}{dt} = \frac{K}{T}e^{-t/T}$$

Analysis of Simple RC Circuit



$$R \cdot i(t) + v(t) = v_T(t)$$

$$i(t) = \frac{d(Cv(t))}{dt} = C\,\frac{dv(t)}{dt}$$

$$\Rightarrow RC\,\frac{dv(t)}{dt} + v(t) = v_T(t)$$

state variable

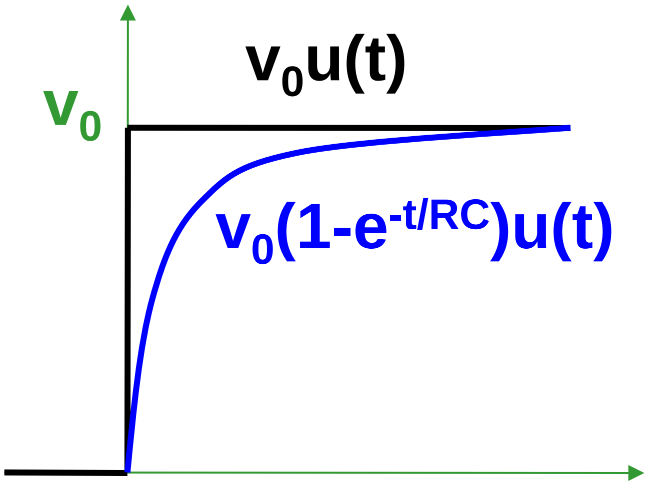
Input waveform

Analysis of Simple RC Circuit

Step-input response:

$$RC\frac{dv(t)}{dt}+v(t)=v_0u(t)$$

$$v(t)=Ke^{-t/RC}+v_0u(t)$$



match initial state:

$$v(0)=0 \quad\Rightarrow\quad K+v_0u(t)=0 \;\Rightarrow\; K+v_0=0$$

output response for step-input:

$$v(t)=v_0(1-e^{-t/RC})u(t)$$

Example 1

Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

Find out

- Time constant T
- D.C Gain K
- Transfer Function
- Step Response

Example 1

The Laplace Transform of Impulse response of a system is actually the transfer function of the system.

Therefore taking Laplace Transform of the impulse response given by following equation.

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S + 0.5} \times 1 = \frac{3}{S + 0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S + 0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S + 1}$$

Example 1

Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

Find out

- Time constant T=2
- D.C Gain K=6
- Transfer Function $\dfrac{C(s)}{R(s)} = \dfrac{6}{2S+1}$
- Step Response

Example 1

For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- **We can find out *C* if initial condition is known e.g.** **$c_s(0)=0$**

$$0 = -6e^{-0.5\times 0} + C$$
$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

Example 1

If initial conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ is a step input, $R(s) = \frac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

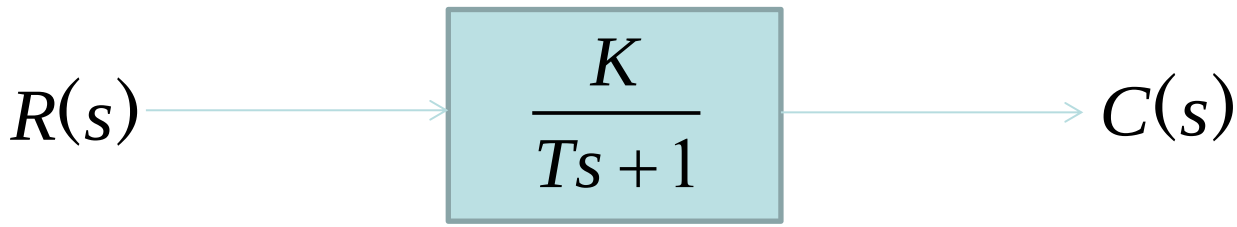
$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$

Ramp Response of 1st Order System

Consider the following 1st order system



$$R(s) = \frac{1}{s^2}$$

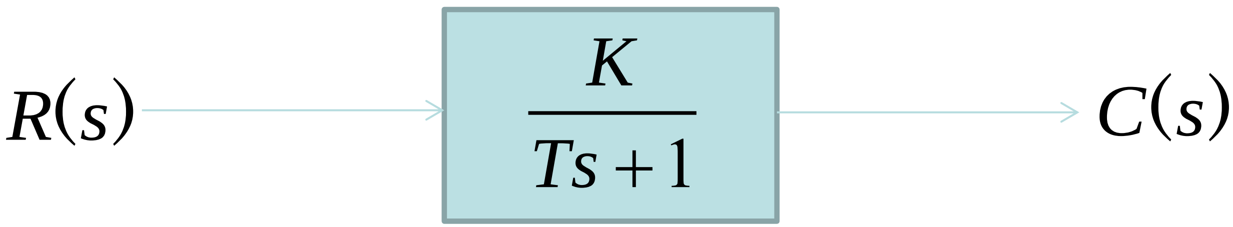
$$C(s) = \frac{K}{s^2(Ts+1)}$$

- **The ramp response is given as**

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$

Parabolic Response of 1st Order System

Consider the following 1st order system



$$R(s) = \frac{1}{s^3} \quad \textbf{Therefore, } C(s) = \frac{K}{s^3 (Ts+1)}$$

Practical Determination of Transfer Function of 1st Order Systems

Often it is not possible or practical to obtain a system's transfer function analytically.

Perhaps the system is closed, and the component parts are not easily identifiable.

The system's step response can lead to a representation even though the inner construction is not known.

With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

Practical Determination of Transfer Function of 1st Order Systems

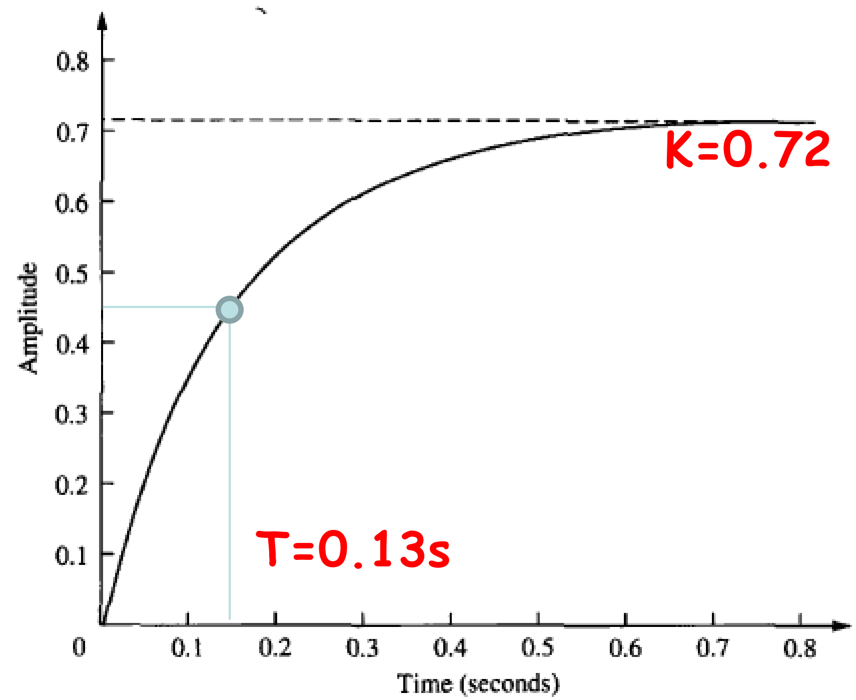
If we can identify T and K empirically we can obtain the transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

Practical Determination of Transfer Function of 1st Order Systems

For example, assume the unit step response given in figure.

- From the response, we can measure the time constant, that is, the time for the amplitude to reach 63% of its final value.
- Since the final value is about 0.72 the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second.
- K is simply steady state value.



- Thus, transfer function is obtained as:

$$\frac{C(s)}{R(s)} = \frac{0.72}{0.13s+1} = \frac{5.5}{s+7.7}$$

First Order System with a Zero

$$\frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

Zero of the system lie at *-1/α* and pole at *-1/T*.

- Step response of the system would be:

$$C(s) = \frac{K(1+\alpha s)}{s(Ts+1)}$$

$$C(s) = \frac{K}{s} + \frac{K(\alpha - T)}{(Ts+1)}$$

$$c(t) = K + \frac{K}{T}(\alpha - T)e^{-t/T}$$

First Order System With Delays

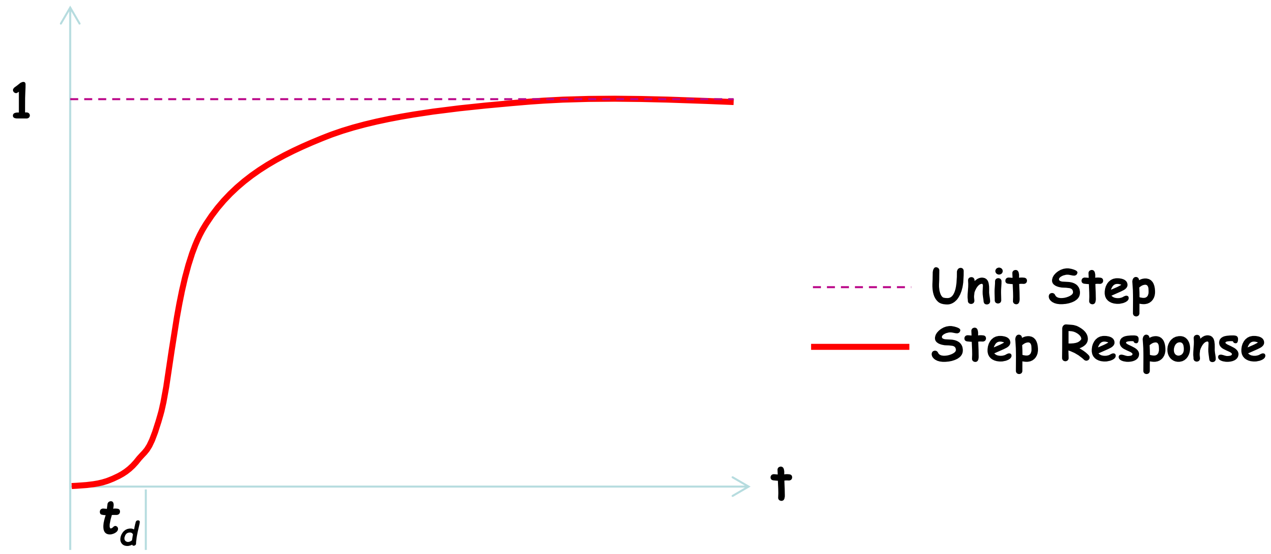
Following transfer function is the generic representation of 1st order system with time lag.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1} e^{-st_d}$$

Where t_d is the delay time.

First Order System With Delays

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1} e^{-st_d}$$



First Order System With Delays

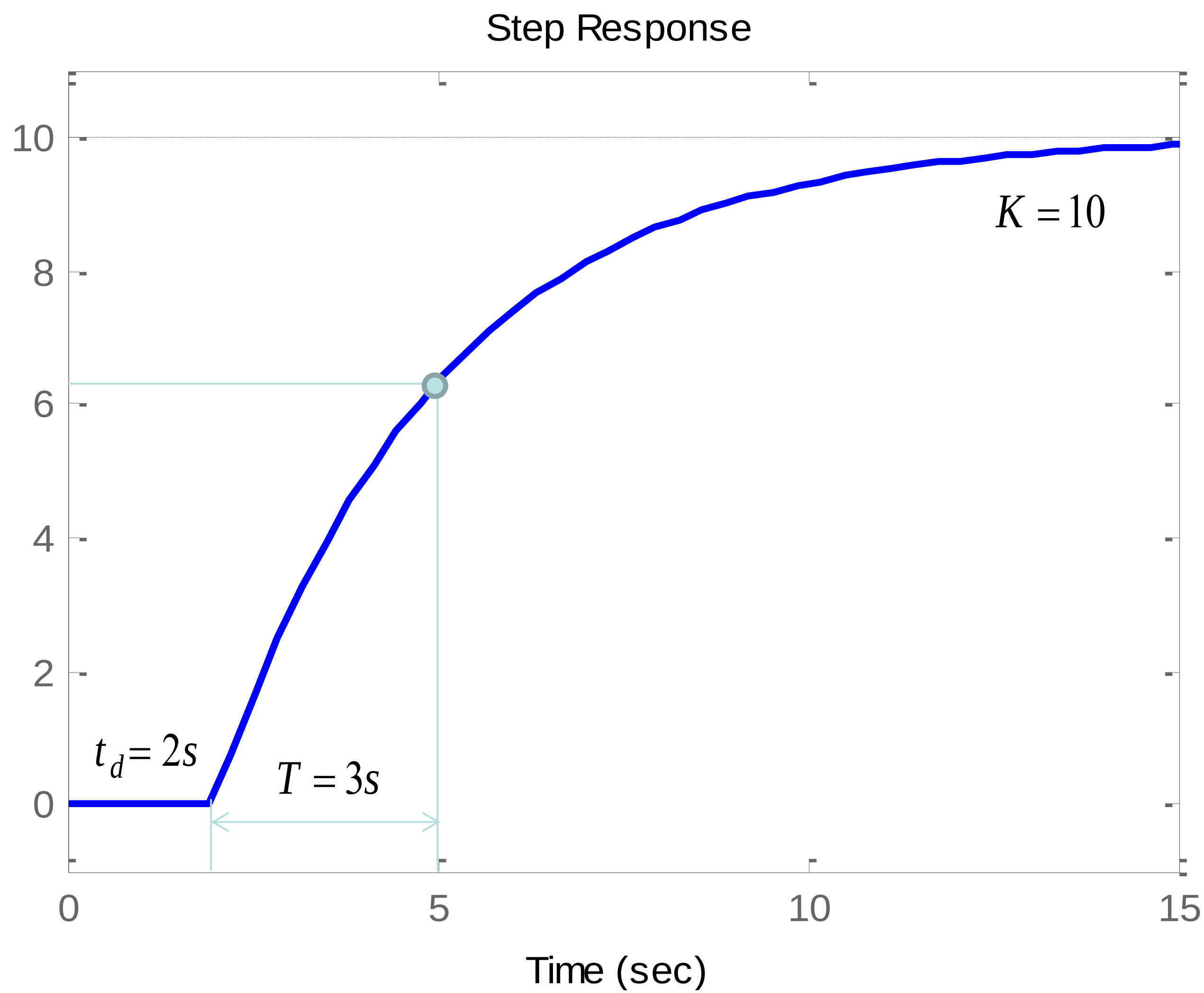
$$\frac{C(s)}{R(s)} = \frac{10}{3s+1}e^{-2s}$$

$$C(s) = \frac{10}{s(3s+1)}e^{-2s}$$

$$L^{-1}[e^{-\partial s}F(s)] = f(t-\partial)u(t-\partial)$$

$$L^{-1}[(\frac{10}{s} + \frac{-10}{s+1/3})e^{-2s}] =$$

$$[10(t-2) - 10e^{-1/3(t-2)}]u(t-2)$$



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